Anisotropic nonperturbative zero modes for passively advected magnetic fields

Alessandra Lanotte[1] and Andrea Mazzino[2,3]

[1]*CNRS, Observatoire de la Côte d'Azur, Boîte Postale 4229, 06304 Nice Cedex 4, France*
[2]*INFM-Dipartimento di Fisica, Università di Genova, I-16142 Genova, Italy*
[3]*The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

An analytic assessment of the role of anisotropic corrections to the isotropic anomalous scaling exponents is given for the d-dimensional kinematic magnetohydrodynamics problem in the presence of a mean magnetic field. The velocity advecting the magnetic field changes very rapidly in time and scales with a positive exponent ξ. Inertial-range anisotropic contributions to the scaling exponents, ζ_j, of second-order magnetic correlations are associated with zero modes and have been calculated nonperturbatively. For $d=3$, the limit $\xi \mapsto 0$ yields $\zeta_j = j - 2 + \xi(2j^3 + j^2 - 5j - 4)/[2(4j^2 - 1)]$, where j ($j \geqslant 2$) is the order in the Legendre polynomial decomposition applied to correlation functions. Conjectures on the fact that anisotropic components cannot change the isotropic threshold to the dynamo effect are also made. [S1063-651X(99)51109-4]

Since Kolmogorov [1] formulated his hypothesis, most of the theories and models in turbulence have used as a key ingredient the restored local isotropy of small-scale structures, even in the presence of large-scale anisotropy. Dealing with such an idealized picture implies that the anisotropic effects, that almost every large-scale forcing indeed involve, are totally disregarded.

Recently, some considerable effort has been made [2–6] to shed some light on the statistics of structure functions when taking anisotropy explicitly into account. When doing this, two major questions emerge: the first concerns the possibility of an *universal* nature of the scaling exponents of the separated isotropic and anisotropic contributions to structure functions; the second concerns the decay of anisotropic fluctuations, and consequently the validity of the local isotropy hypothesis. The state of the art on this subject, especially when looking at experimental data, does not give unique answers.

Here, our aim is to analyze, through nonperturbative calculations, the effects of anisotropy on anomalous (i.e., non-dimensional) scaling exponents of the magnetic field correlations, within a kinematic magnetohydrodynamics (MHD) problem (i.e., when reaction of the magnetic field on the velocity field is neglected). Since the advecting velocity field that we consider is δ-correlated in time, an analytical approach is possible: the main result shown in this Rapid Communication is that the anomalous scaling exponent, ζ_0, associated with the isotropic contribution, is dominant with respect to the anisotropic ones, ζ_j. In addition, the entire set of anisotropic scaling exponents, ζ_j, is given, showing the existence of a hierarchy related to the degree of anisotropy j, such that $\zeta_0 < \zeta_2 < \cdots$. This result is the signature of the emergence of local isotropy at small scales. Such hierarchy relations are analogous to those found in Ref. [2], where the scalar advection in the presence of large-scale anisotropy is studied exploiting the field theoretic renormalization group (RG).

We also remark that the scenario outlined here is compatible with those arising from the results shown in Refs. [4,6] where, respectively, experimental and direct numerical simu-

lation data of anisotropic turbulence have been analyzed. Considering the kinematic MHD problem, the issue of the threshold for the appearance on the unbounded growth of the magnetic field is also present. We shall report a conjecture according to which we can argue that the anisotropic components do not play any effect in this sense.

In the presence of a mean component $\boldsymbol{B}^o$ (actually supposed varying on very large scales $\sim L$, the largest one in our problem) the kinematic MHD equations describing the evolution of the fluctuating part, $\boldsymbol{B}$, of the magnetic field are [7]

$$\partial_t B_\alpha + \boldsymbol{v} \cdot \boldsymbol{\partial} B_\alpha = \boldsymbol{B} \cdot \boldsymbol{\partial} v_\alpha + \boldsymbol{B}^o \cdot \boldsymbol{\partial} v_\alpha + \kappa \partial^2 B_\alpha,$$

$$\alpha = 1, \ldots, d, \tag{1}$$

where the velocity, $\boldsymbol{v}$, is a zero-mean Gaussian random process, homogeneous, isotropic and white-in-time, κ is the magnetic diffusivity, and d is the dimension of the space. Both $\boldsymbol{v}$ and $\boldsymbol{B}$ are divergence-free fields. The term $\boldsymbol{B}^o \cdot \boldsymbol{\partial} v_\alpha$ in Eq. (1) plays the same role as an external forcing driving the system and being also a source of anisotropy for the magnetic field statistics.

The velocity is self-similar with the two-point correlation function,

$$\langle v_\alpha(\boldsymbol{r},t) v_\beta(\boldsymbol{r}',t') \rangle = \delta(t-t')[d^0_{\alpha\beta} - S_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}')], \tag{2}$$

where $S_{\alpha\beta}(\boldsymbol{r})$ is fixed by isotropy and scaling behavior, and scales with the exponent ξ, in the range $0 < \xi < 2$:

$$S_{\alpha\beta}(\boldsymbol{r}) = Dr^\xi \left[(d+\xi-1)\delta_{\alpha\beta} - \xi \frac{r_\alpha r_\beta}{r^2} \right]. \tag{3}$$

The δ-correlation in time of $\boldsymbol{v}$ permits to exploit the Gaussian integration by parts (a comprehensive description of this technique can be found, e.g., in Ref. [8]) to obtain closed, exact equations for $C_{\alpha\beta}(\boldsymbol{r},t) \equiv \langle B_\alpha(\boldsymbol{x},t)B_\beta(\boldsymbol{x}+\boldsymbol{r},t) \rangle$. After some manipulations of Eq. (1), such equations read

$$\partial_t C_{\alpha\beta} = S_{ij}\partial_i\partial_j C_{\alpha\beta} - (\partial_j S_{i\beta})\partial_i C_{\alpha j} - (\partial_j S_{\alpha i})(\partial_i C_{j\beta})$$
$$+ (\partial_i\partial_j S_{\alpha\beta})(C_{ij} + B_i^o B_j^o) + 2\kappa\partial^2 C_{\alpha\beta}. \qquad (4)$$

A further equation for $C_{\alpha\beta}$ follows from the solenoidal condition on $\boldsymbol{B}$,

$$\partial_\alpha C_{\alpha\beta} = 0. \qquad (5)$$

For what follows, it is worth emphasizing two properties of $C_{\alpha\beta}$:

(i) because of homogeneity, $C_{\alpha\beta}$ is left invariant under the following set of transformations:

$$\boldsymbol{r}\mapsto -\boldsymbol{r} \quad \text{and} \quad \alpha\leftrightarrow\beta; \qquad (6)$$

(ii) $C_{\alpha\beta}(\boldsymbol{r}) = C_{\alpha\beta}(-\boldsymbol{r})$, as it follows from Eq. (4) after the substitution $\boldsymbol{r}\mapsto -\boldsymbol{r}$.

As shown in Ref. [9], in the isotropic case (i.e., $\boldsymbol{B}^o = \boldsymbol{0}$ in our problem) anomalies appear already in the scaling exponents of the second-order magnetic correlations, $C_{\alpha\beta}$, and have been calculated nonperturbatively by the author. Anomalous scaling laws are associated with zero modes of the closed equations satisfied by the equal-time correlation functions. In Ref. [10], anomalous exponents for higher-order correlations have been calculated to the order ξ by exploiting the RG technique.

The extraction of anisotropic contributions to the isotropic scaling of $C_{\alpha\beta}(\boldsymbol{r})$ found in Ref. [9], and the investigation of their effect (if any) on the emergence of the dynamo effect are the main questions addressed in the present paper. The main technical difference with respect to Ref. [9] is that the angular structure of zero modes has now to be explicitly taken into account.

In the presence of anisotropy, the most general expression for the two-point magnetic correlations, $C_{\alpha\beta}(\boldsymbol{r})$, in the stationary state involves five (two in the isotropic case) functions depending on both $r \equiv |\boldsymbol{x} - \boldsymbol{x}'|$ and $z \equiv \cos\theta = \hat{\boldsymbol{B}}^o \cdot \boldsymbol{r}/r$, where $\hat{\boldsymbol{B}}^o$ is the unit vector corresponding to the direction selected by the mean magnetic field. We remark that the space is anisotropic but still homogeneous, so there is no explicit dependence on the points $\boldsymbol{x}, \boldsymbol{x}'$, but only on their distance. Namely,

$$C_{\alpha\beta}(\boldsymbol{r}) = \mathcal{F}_1(r,z)\delta_{\alpha\beta} + \mathcal{F}_2(r,z)\frac{r_\alpha r_\beta}{r^2} + \mathcal{F}_3(r,z)\frac{\hat{B}_\alpha^o r_\beta}{r}$$

$$+ \mathcal{F}_4(r,z)\frac{\hat{B}_\beta^o r_\alpha}{r} + \mathcal{F}_5(r,z)\hat{B}_\alpha^o\hat{B}_\beta^o. \qquad (7)$$

From the properties (i) and (ii) of $C_{\alpha\beta}(\boldsymbol{r})$ one immediately obtains the following relations for the $\mathcal{F}$'s:

$$\mathcal{F}_i(r,z) = \mathcal{F}_i(r,-z), \quad i = 1,2,5, \qquad (8)$$

$$\mathcal{F}_3(r,z) = -\mathcal{F}_3(r,-z), \qquad (9)$$

$$\mathcal{F}_3(r,z) = \mathcal{F}_4(r,z). \qquad (10)$$

Substituting the expression (7) into Eq. (4) and using the chain rules, we obtain, after lengthy but straightforward al-

gebra, the following four equations (corresponding to the projections over $r_\alpha r_\beta/r^2$, $\delta_{\alpha\beta}$, $\hat{B}_\alpha^o r_\beta/r$ and $\hat{B}_\alpha^o\hat{B}_\beta^o$):

$$[a_1 r^2\partial_r^2 + b_1 r\partial_r + c_1(1-z^2)\partial_z^2 + d_1 z\partial_z + e_1]\mathcal{F}_1$$
$$+ [f_1 r\partial_r + g_1 z\partial_z + j_1]\mathcal{F}_2 + [k_1 zr\partial_r + l_1 z^2\partial_z + m_1 z$$
$$+ n_1\partial_z]\mathcal{F}_3 + [o_1 + p_1 z^2]\mathcal{F}_5 = (q_1 + r_1 z^2)B^{o2}, \qquad (11)$$

$$a_2\mathcal{F}_1 + [b_2 r^2\partial_r^2 + c_2 r\partial_r + d_2(1-z^2)\partial_z^2 + e_2 z\partial_z + f_2]\mathcal{F}_2$$
$$+ g_2 z\mathcal{F}_3 + [k_2 + l_2 z^2]\mathcal{F}_5 = (m_2 + n_2 z^2)B^{o2}, \qquad (12)$$

$$a_3\partial_z\mathcal{F}_1 + b_3\partial_z\mathcal{F}_2 + [c_3 r^2\partial_r^2 + d_3 r\partial_r + e_3(1-z^2)\partial_z^2$$
$$+ f_3 z\partial_z + g_3]\mathcal{F}_3 + [j_3 zr\partial_r + (k_3 + l_3 z^2)\partial_z + m_3 z]\mathcal{F}_5$$
$$= n_3 B^{o2}z, \qquad (13)$$

$$a_4\partial_z\mathcal{F}_3 + [b_4 r^2\partial_r^2 + c_4 r\partial_r + d_4(1-z^2)\partial_z^2 + e_4 z\partial_z + f_4]\mathcal{F}_5$$
$$= g_4 B^{o2}, \qquad (14)$$

where the coefficients $a_i, b_i, \ldots, r_i$ are cumbersome functions of ξ and d and will not be here reported for the sake of brevity. Without loss of generality, we have fixed $D = 1$ in Eq. (3), and we have neglected all terms involving the magnetic diffusivity κ, our attention being indeed focused in the inertial range of scales, i.e., $\eta \ll r \ll L$, where $\eta = \kappa^{1/\xi}$ is the dissipative scale for the problem.

With the substitution of expression (7), the solenoidal condition (5) splits into the following couple of equations:

$$[r\partial_r + (d-1)]\mathcal{F}_1 + [r\partial_r - z\partial_z]\mathcal{F}_2 + [zr\partial_r + \partial_z - z^2\partial_z - z]\mathcal{F}_3$$
$$= 0, \qquad (15)$$

$$\partial_z\mathcal{F}_2 + [r\partial_r + d]\mathcal{F}_3 + [zr\partial_r + (1-z^2)\partial_z]\mathcal{F}_5 = 0 \qquad (16)$$

associated to the projections over r_β/r and $\hat{B}_\beta^o$, respectively.

From relation (10) and Eqs. (15) and (16) it then follows that only two functions, the $\mathcal{F}$s, in Eq. (7), are independent.

According to the old idea of Kolmogorov, in cascadelike mechanisms of transfer of energy towards small scales, anisotropy present at the integral scale should eventually decay during the (chaotic) transfer. One could thus argue that, at least at small scales, anisotropic corrections to the isotropic contribution would be smaller and smaller as the order of anisotropic contributions increases. For Navier-Stokes turbulence in channel flow, such a picture has recently been confirmed by Arad et al. [6].

As we shall see, the above physical hint can be easily exploited if one decomposes functions $\mathcal{F}$ on the Legendre polynomial basis. Accordingly, we have

$$\mathcal{F}_i(r,z) = \sum_{l=0}^\infty f_{2l}^{(i)}(r)P_{2l}(z), \quad i = 1,2,5, \qquad (17)$$

$$\mathcal{F}_3(r,z) = \sum_{l=0}^\infty f_{2l+1}^{(3)}(r)P_{2l+1}(z), \qquad (18)$$

where the separation of even and odd orders in Eqs. (17) and (18) arises as a consequence of the symmetries expressed by

relations (8) and (9), respectively. As larger l's correspond to higher order anisotropic contributions, we thus expect that, when scaling behavior sets in (i.e., for $\eta \ll r \ll L$), we shall have

$$f_l^{(i)}(r) \propto r^{\zeta_l^{(i)}}, \quad \text{with} \quad \zeta_0^{(i)} < \zeta_1^{(i)} < \cdots. \quad (19)$$

We would like to obtain equations for $f_l^{(i)}(r)$, to then be solved for the $\zeta_l^{(i)}$. To do this, in Eqs. (11)–(16) we have to express quantities such as $z^j \partial_z^m$ ($j,m=0,1,2$, with $j \neq 1$ and $m \neq 2$) in terms of Legendre polynomials. Recalling to this purpose well-known relations involving these latter (see, e.g., Ref. [11]), we obtain general expressions, as e.g.,

$$\partial_z \mathcal{F}_i(r,z) = \sum_{l=0}^{\infty} P_l(z) \left[(2l+1) \sum_{q=0}^{\infty} f_{2q+l+1}^{(i)}(r) \right], \quad (20)$$

from which we notice that an arbitrary l-order anisotropic contribution is coupled to all larger orders. The resulting equations arising from Eqs. (11)–(16) are thus not closed.

Closed equations for the f's can actually be obtained by exploiting Eq. (19), i.e. by using the hypothesis of a hierarchy in the self-similar behavior of the f's. Accordingly, in Eqs. (11)–(16) at a given order j, for each function $f_l^{(i)}$ we need to retain only its lower order contributions, with $l \leq j$. It is worth noticing that we can control the validity of this (physical) assumption in a self-consistent way, at the end of our calculation.

As a result, one obtains the following (infinite) set of closed differential equations, valid for j even:

$$a_1 r^2 f_j''^{(1)} + b_1 r f_j'^{(1)} + c_1 f_j^{(1)} + d_1 r f_j'^{(2)} + e_1 f_j^{(2)} + g_1 r f_{j-1}'^{(3)}$$
$$+ j_1 f_{j-1}^{(3)} + k_1 f_{j-2}^{(5)} = B^{o2} l_1 \delta_{j2}, \quad (21)$$

$$a_2 f_j^{(1)} + b_2 r^2 f_j''^{(2)} + c_2 r f_j'^{(2)} + d_2 f_j^{(2)} + e_2 f_{j-1}^{(3)} + g_2 f_{j-2}^{(5)}$$
$$= B^{o2} j_2 \delta_{j2}, \quad (22)$$

$$a_3 f_j^{(1)} + b_3 f_j^{(2)} + c_3 r^2 f_{j-1}''^{(3)} + d_3 r f_{j-1}'^{(3)} + e_3 f_{j-1}^{(3)} + g_3 r f_{j-2}'^{(5)}$$
$$+ j_3 f_{j-2}^{(5)} = B^{o2} k_1 \delta_{j2}, \quad (23)$$

$$a_4 f_{j-1}^{(3)} + b_4 r^2 f_{j-2}''^{(5)} + c_4 r f_{j-2}'^{(5)} + d_4 f_{j-2}^{(5)} = B^{o2} e_4 \delta_{j2}, \quad (24)$$

$$r f_j'^{(1)} + (d-1) f_j^{(1)} + r f_j'^{(2)} - j f_j^{(2)} + \frac{j}{2j-1} r f_{j-1}'^{(3)}$$
$$- \frac{j^2}{2j-1} f_{j-1}^{(3)} = 0, \quad (25)$$

$$(2j-1) f_j^{(2)} + r f_{j-1}'^{(3)} + d f_{j-1}^{(3)} + r \frac{j-1}{2j-3} f_{j-2}'^{(5)}$$
$$- \frac{(j-2)(j-1)}{2j-3} f_{j-2}^{(5)} = 0, \quad (26)$$

where coefficients $a_i, b_i, \cdots$ [different from those defined in Eqs. (11)–(14)] depend only on ξ and d. For $j=0$, the coefficients relative to $f_{j-1}^{(3)}$ and $f_{j-2}^{(5)}$ (and their derivatives) are zero and the resulting equations for $f_0^{(1)}$ and $f_0^{(2)}$ are exactly

as in Ref. [9]. The structure of the above equations fixes the relation between the scaling exponents relative to different f's. Indeed, when searching for power law solutions $f_j^{(i)}(r) \propto r^{\zeta_j^{(i)}}$, in order to obtain balanced equations the "oblique" relations must hold

$$\zeta_j \equiv \zeta_j^{(1)} = \zeta_j^{(2)} = \zeta_{j-1}^{(3)} = \zeta_{j-2}^{(5)}. \quad (27)$$

We are now ready to show that nontrivial scaling behaviors for the f's take place due to zero modes, i.e., the solutions of the homogeneous problem associated with Eqs. (21)–(26). To that purpose, let us consider such differential problem with no forcing (i.e., $B^o = 0$). If, when looking for power law solutions, we exploit Eq. (27) and the fact that only two functions of the f's are independent, our differential problem is mapped into an algebraic one. In the latter the emergence of zero modes reduces to imposing the existence of nonzero solutions of a 2×2 homogeneous linear system. The calculation is lengthy but straightforward, and the following expressions for the zero-mode exponents are obtained [12]:

$$\zeta_j^{\pm} = -\frac{1}{2(d-1)} \{ 2\xi + d^2 - d - [-2d^3\xi - 2d^2\xi^2 - 6d^3$$
$$+ 4\xi^2 d + 8 + 10d\xi + 20dj - 20d - 8\xi - 8j + 4d^2 j^2$$
$$+ 2\xi^2 - 4\xi j^2 + 17d^2 - 8dj^2 + 8\xi j + 4d^3 j + 4d^2 j\xi$$
$$+ 4dj^2\xi + 4j^2 - 16d^2 j - 12d\xi j + d^4 \pm 2\sqrt{K}(d-1)$$
$$\times (2-\xi)]^{1/2} \}, \quad (28)$$

where

$$K = (d-1)(d^3 + 4d^2 j - 5d^2 + 2d^2\xi + \xi^2 d + 4d\xi j - 6d\xi + 8d$$
$$- 12dj + 4dj^2 - \xi^2 + 4\xi + 8j - 8\xi j - 4 - 4j^2 + 4\xi j^2).$$

Some remarks are in order. First, ζ_0^+ coincides with the isotropic solution obtained by Vergassola in Ref. [9], the admissibility of which has been proved by the author. Second, ζ_0^- diverges as r^{-1} at the dissipative scale η. The exponent ζ_0^- is thus not admissible. Third, $\zeta_j^+ > \zeta_j^-$ for all j. This means that, in the inertial range of scales (i.e., $r/L \ll 1$) the leading zero-mode solutions for $j>2$ are associated with ζ_j^-. We can thus define the leading set of zero modes ζ_j as $\zeta_0 \equiv \zeta_0^+$; $\zeta_j \equiv \zeta_j^-$ for $j \geq 2$. In particular, for $j=2$, the asymptotic limits $\xi \ll 1$ and $d \gg 1$ are, respectively,

$$\zeta_2 = \frac{2\xi}{(d-1)(d+2)} + O(\xi^2); \quad \zeta_2 = \frac{2\xi}{d^2} + O(1/d^3). \quad (29)$$

Let us briefly discuss the infrared (IR) behavior of ζ_j. In the absence of forcing terms, there is no way to satisfy the IR boundary condition [i.e., $C_{\alpha\beta}(r) = 0$ for $r \mapsto \infty$]: zero mode associated with ζ_j indeed diverges for $r \mapsto \infty$. As a consequence, zero modes for $j \geq 2$ are not globally acceptable. For $j=2$ the situation changes completely: in this case, Eqs. (21)–(24) are forced and, as in Ref. [9], IR boundary conditions can be satisfied by matching at the large scale L zero-mode solutions with those of the inhomogeneous problem. From the above considerations, it also follows that zero
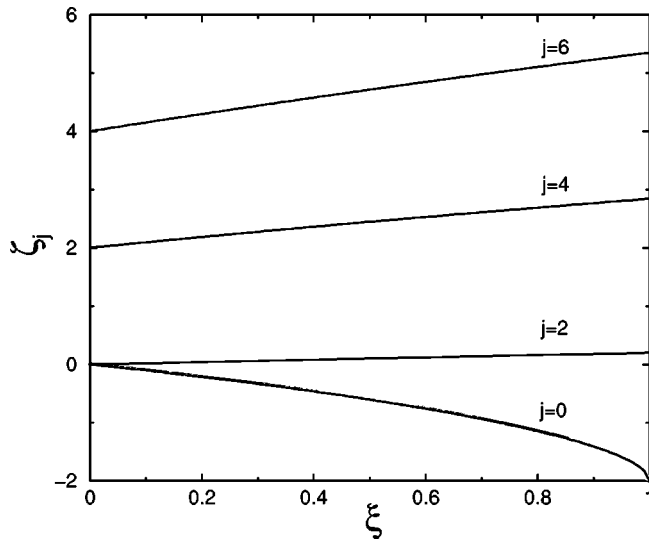
FIG. 1. Behavior of ζ_j vs ξ for $d = 3$ and (from below to above) $j = 0$, $j = 2$, 4, and 6.

modes associated with ζ_j become acceptable for all orders j when a fully anisotropic forcing term (i.e., projecting on all Legendre polynomials) is added in the right-hand side of Eq. (1).

Finally, autoconsistency of our solution for ζ_j, that is, the validity of the hierarchy in Eq. (19), can be immediately checked from Fig. 1, where the behaviors of a few ζ_j as functions of ξ are shown in the three-dimensional case for $j = 0$, 2, 4, and 6 (from below to above). It is easy to verify that the increasing of scaling exponents with j actually holds for all values of j and d.

The expression for ζ_j allows us to make some conjectures on the role played by anisotropic effects on the emergence of the dynamo effect. It is known [9,13] that in the isotropic case an unbounded growth of the magnetic field (dynamo effect) arises for $\xi > 1$. The question addressed here is whether anisotropic component can contribute to destabilize the system, shifting toward smaller values of ξ the threshold for the dynamo. We note that, in the isotropic case, dynamo arises when the exponent related to the admissible zero mode becomes complex. In this case, zero-mode solutions have sinusoidal components, a fact that makes possible their matching with the appropriate boundary conditions also in the absence of forcing (i.e., the system is self-maintained). This happens for $\xi > 1$, $\xi = 1$ being the threshold. Taking such condition as a criterion to select the emergence of an unbounded growth, we can conclude that there is no effect played by the anisotropic components. Indeed, it is easily verified from Eq. (28) that, for all d's, ζ_j is real for $\xi \in [0,1]$.

In conclusion, we have presented a system where the extraction of anisotropic contributions to the anomalous scaling of the equal-time magnetic correlation functions can be performed in a nonperturbative way. We have calculated the entire set of universal anomalous exponents, ζ_j, and we have given an analytic assessment of the dominance of the fundamental exponent associated with the isotropic shell. More generally, the hierarchy $\zeta_0 < \zeta_2 < \cdots \zeta_j < \cdots$ has been proved. The picture here drawn is in agreement with recent findings by Antonov [2], where the passive scalar problem is studied, and by Arad et al. [6] for Navier–Stokes turbulence.

[1] A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. URSS **30**, 301 (1941).

[2] N. V. Antonov, e-print archive chao-dyn/9808011.

[3] S. G. Saddoughi and S. V. Veeravalli, J. Fluid Mech. **268**, 333 (1994).

[4] V. Borue and S. A. Orszag, J. Fluid Mech. **306**, 293 (1994).

[5] I. Arad, B. Dhruva, S. Kurien, V. S. L'vov, I. Procaccia, and K. R. Sreenivasan, Phys. Rev. Lett. **81**, 5330 (1998).

[6] I. Arad, L. Biferale, I. Mazzitelli, and I. Procaccia, Phys. Rev. Lett. **82**, 5040 (1999).

[7] Ya.B. Zeldovich, A.A. Ruzmaikin, and D.D. Sokoloff, *Magnetic fields in Astrophysics* (Gordon and Breach, New York, 1983).

[8] U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).

[9] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).

[10] L. Ts. Adzhemyan and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998).

[11] I. S. Gradshtejn and I.M. Ryzhik, *Table of Integrals Series and Products* (Academic Press, New York, 1965).

[12] We are particularly grateful to N.V. Antonov, I. Arad, L. Biferale, and I. Procaccia to have found an error in a previous expression for ζ_j.

[13] A. P. Kazantzev, Zh. Eksp. Teor. Fiz. **53**, 1806 (1967) [Sov. Phys. JETP **26**, 1031 (1968)].